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LETTER TO THE EDITOR

Two ways for Hopf bifurcation with symmetry

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Abstract. Two cases are presented of bifurcation problems in the presence of a symmetry: it is shown that suitable group theoretical assumptions lead to the existence of Hopf bifurcation.

The problem of the Hopf bifurcation in the presence of a symmetry has already been considered in great detail in recent literature (we quote only the papers [1, 2]; see also references therein). There are, however, some further possibilities for the appearance of a bifurcation of Hopf type, which do not seem to be fully explored; we refer mainly to case 1 below; case 2 in fact is essentially covered in [1], and it is briefly mentioned here for comparison and for completeness.

Let us consider the problem of finding periodic bifurcating solutions of the equation

$$\dot{u} = f(\lambda, u) \qquad f: R \times R^N \to R^N \qquad f(\lambda, 0) = 0 \tag{1}$$

where $u \in R^N$, $u = u(t)$, $\lambda \in R$, and with the usual regularity assumptions. Assume now that this problem is covariant with respect to a symmetry group G, acting on R^N through a real representation T:

$$f(\lambda, T(g)u) = T(g)f(\lambda, u) \qquad \forall g \in \mathrm{G} \tag{2}$$

and assume in particular the following.

Case 1. T is irreducible if considered as a real representation, but, by complexification of the space, it splits into the direct sum of two irreducible complex conjugated representations:

$$T \simeq D \oplus \bar{D} \tag{3}$$

(with D inequivalent to $\bar{D}$, otherwise one could be led to the case of a periodic 'quaternionic' bifurcation; see [3, 4]). Explicitly, one has

$$T = \tfrac{1}{2}\begin{pmatrix} I_n & -\mathrm{i}I_n \\ -\mathrm{i}I_n & I_n \end{pmatrix}(D \oplus \bar{D})\begin{pmatrix} I_n & \mathrm{i}I_n \\ \mathrm{i}I_n & I_n \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, D & -\mathrm{Im}\, D \\ \mathrm{Im}\, D & \mathrm{Re}\, D \end{pmatrix}$$

where I_n is the n-dimensional unit matrix ($N = 2n$). This implies that there are two independent operators which commute with T, namely the N-dimensional unit I_N, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

In particular, then, the linearised part $L(\lambda)=f'_u(\lambda, 0)$ has the form

$$L(\lambda)=a(\lambda)I_N+b(\lambda)J. \tag{4}$$

Therefore, this is an example for case (b) of § 2 in the paper [1], a case which was not examined in that reference.

For concreteness, we will assume in this case that G is a unitary group. Let us now suppose that in the basis space C^n of D there is a vector ξ such that:

(a) denoting by H the isotropy subgroup of ξ, i.e.

$$D(h)\xi=\xi \qquad \forall h\in \mathrm{H}$$

the fixed point subspace of H, i.e. the set of vectors in C^n which are left fixed by H, is one-dimensional (in a complex sense) and then is spanned only by ξ.

This generalises a typical assumption [5] which, as is well known (see [1] and references therein), was at the basis of some theorems concerning bifurcation with symmetry. Using a similar argument, assumption (a), together with covariance property (2), implies that, putting

$$\xi=x'+\mathrm{i}x'' \qquad \text{and} \qquad x\equiv(x', x'')$$

the problem (1) can be restricted to the two-dimensional real subspace X spanned by x' and x'':

$$\dot{x}=f(\lambda, x) \qquad f: R\times X\to X \tag{5}$$

having denoted again by f its restriction to $R\times X$. The important point is now that the original symmetry G induces on X an SO_2 covariance: in fact, the action of G on X will be

$$\xi\to\xi\,\mathrm{e}^{\mathrm{i}m\phi} \qquad \text{and} \qquad \bar{\xi}\to\bar{\xi}\,\mathrm{e}^{-\mathrm{i}m\phi} \qquad (m=0, 1, 2, \ldots) \tag{6}$$

and therefore

$$\begin{pmatrix} x' \\ x'' \end{pmatrix}\to\begin{pmatrix} \cos m\phi & -\sin m\phi \\ \sin m\phi & \cos m\phi \end{pmatrix}\begin{pmatrix} x' \\ x'' \end{pmatrix}. \tag{6'}$$

Then, apart from the case $m=0$, (5) turns out to be covariant with respect to the SO_2 symmetry (6'). It can be noted, of course, that in addition to this 'external' or 'spatial' covariance, as explained in detail in [1], our (5) (and also (1)) exhibits a different type of SO_2 'temporal' covariance, which is intrinsically induced by time 'translations' $t\to t+s$ (mod T, the period of functions $x(t)$), and acts according to the SO_2 representation $D(s)x(t)=x(t+s)$. In conclusion, one recovers a special type of Hopf problem, due to this (spatial) rotational symmetry of (5). Now, standard hypotheses on the functions $a(\lambda)$, $b(\lambda)$ in (4) (e.g. $a(\lambda_0)=0$, $b(\lambda_0)\neq 0$, $\partial a(\lambda_0)/\partial\lambda\neq 0$) will directly ensure the appearance of a bifurcated periodic solution of the problem (1), lying in the subspace X.

As an example, let $G=SU_3$: then, if D is its fundamental three-dimensional complex representation acting on the 'quarks' ψ^α ($\alpha=1, 2, 3$), one can choose $\xi=\psi^1$ and then m in (6) is equal to 1. If instead D is the six-dimensional complex representation acting on the second-order symmetric tensors $\psi^{\alpha\beta}$, one can choose $\xi=\psi^{11}$, and then $m=2$.

Case 2. Suppose now that T is reducible also in the real space R^N, and splits into the direct sum of two real irreducible equivalent representations

$$T = D \oplus D' \qquad D \simeq D'. \tag{7}$$

In this case, the linear part of $f(\lambda, u)$ has the form $(N = 2n)$

$$L(\lambda) = \begin{pmatrix} a_1(\lambda)I_n & b_1(\lambda)I_n \\ b_2(\lambda)I_n & a_2(\lambda)I_n \end{pmatrix}. \tag{8}$$

Assuming that there is in the real basis space of D a vector x_1 (and then a vector x_2 for D') satisfying the property (a) in which the word 'complex' is now changed to 'real' (cf [5]), then the problem (1) can be restricted to the two-dimensional real space X spanned by x_1, x_2, just as in case 1, with the main difference that here no spatial rotational symmetry is present. In this form, the problem becomes a completely standard Hopf bifurcation problem.

Clearly, for any solution of both cases 1 and 2, one can construct families ('orbits') of equivalent solutions by applying the group transformations $T(g)$, $g \in G$.

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